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Integer-valued polynomials on algebras

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ABSTRACT

Let D be a domain with quotient field K and A a D-algebra. A polynomial with coefficients in K that maps every element of A to an element of A is called integer-valued on A. For commutative A we also consider integer-valued polynomials in several variables. For an arbitrary domain D and I an arbitrary ideal of D we show I-adic continuity of integer-valued polynomials on A. For Noetherian one-dimensional D, we determine spectrum and Krull dimension of the ring $\mathrm{Int}_D(A)$ of integer-valued polynomials on A. We do the same for the ring of polynomials with coefficients in $M_n(K)$, the K-algebra of $n \times n$ matrices, that map every matrix in $M_n(D)$ to a matrix in $M_n(D)$.

1. Introduction

Let D be a domain with quotient field K and A a D-algebra, such as, for instance, a group ring $D(G)$ or the matrix algebra $M_n(D)$.

We are interested in the rings of polynomials

$$\mathrm{Int}_D(A) = \{ f \in K[x] \mid f(A) \subseteq A \},$$

and, if A is commutative,

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$$\mathrm{Int}_D^n(A) = \{ f \in K[x_1, \ldots, x_n] \mid f(A^n) \subseteq A \}.$$

Elements of the D-algebra A are plugged into polynomials with coefficients in K via the canonical homomorphism $\iota_A : A \to K \otimes_D A$, $\iota_A(a) = 1 \otimes a$.

In the special case $A = D$ these rings are known as rings of integer-valued polynomials, cf. [3]. They provide natural examples of non-Noetherian Prüfer rings [5,11], and have been used for proving results on the n-generator property in Prüfer rings [2]. Also, integer-valued polynomials are useful for polynomial interpolation of functions from D to D [8,4], and satisfy other interesting algebraic conditions such as analogues of Hilbert's Nullstellensatz [3,9].

These desirable properties of rings of integer-valued polynomials have motivated the generalization to polynomials with coefficients in K acting on a D-algebra A [10,12]. So far, not much is known about rings of integer-valued polynomials on algebras. We know that they behave somewhat like the classical rings of integer-valued polynomials if the D-algebra A is commutative. For instance, Loper and Werner [12] have shown that $\mathrm{Int}_{\mathbb{Z}}(\mathcal{O}_K)$ is Prüfer. If A is non-commutative, however, the situation is radically different. For instance, $\mathrm{Int}_{\mathbb{Z}}(M_2(\mathbb{Z}))$ is not Prüfer [12], and is far from allowing interpolation [10].

We will describe the spectrum of $\mathrm{Int}_D(A)$, for a one-dimensional Noetherian ring D and a finitely generated torsion-free D-algebra A, in the hope that this will facilitate further research. We will investigate more closely the special case of $A = \mathrm{M}_n(D)$: we determine a polynomially dense subset of $\mathrm{M}_n(D)$ and describe the image of a given matrix under the ring $\mathrm{Int}_D(\mathrm{M}_n(D))$.

A different ring of integer-valued polynomials on the matrix algebra $M_n(D)$, consisting of polynomials with coefficients in $M_n(K)$ that map matrices in $M_n(D)$ to matrices in $M_n(D)$, has been introduced by Werner [13]. We will show that it is isomorphic to the algebra of $n \times n$ matrices over "our" ring $\mathrm{Int}_D(\mathrm{M}_n(D))$ of integer-valued polynomials on $M_n(D)$ with coefficients in K.

Before we give a precise definition of the kind of D-algebra A for which we will investigate $\mathrm{Int}_D(A)$, a few examples. D is always a domain with quotient field K, and not a field.

1.1. Example. For fixed $n \in \mathbb{N}$, let $A = \mathrm{M}_n(D)$ be the D-algebra of $n \times n$ matrices with entries in D and

$$\mathrm{Int}_D(\mathrm{M}_n(D)) = \{ f \in K[x] \mid \forall C \in \mathrm{M}_n(D)\colon\ f(C) \in \mathrm{M}_n(D) \}.$$

1.2. Example. Let $\mathrm{H} = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ be the $\mathbb{Q}$-algebra of rational quaternions, $\mathrm{L} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ the $\mathbb{Z}$-subalgebra of Lipschitz quaternions, and

$$\mathrm{Int}_{\mathbb{Z}}(\mathrm{L}) = \{ f \in \mathbb{Q}[x] \mid \forall z \in \mathrm{L}\colon\ f(z) \in \mathrm{L} \}.$$

1.3. Example. Let G be a finite group, $K(G)$ and $D(G)$ the respective group rings, and

$$\mathrm{Int}_D(D(G)) = \{ f \in K[x] \mid \forall z \in D(G)\colon\ f(z) \in D(G) \}.$$

If G is commutative, we also consider

$$\mathrm{Int}_D^n(D(G)) = \{ f \in K[x_1, \ldots, x_n] \mid \forall z \in D(G)^n\colon\ f(z) \in D(G) \},$$

for $n \in \mathbb{N}$, where $D(G)^n = D(G) \times \cdots \times D(G)$ denotes the Cartesian product of n copies of $D(G)$.

1.4. Example. Let $D \subseteq A$ be Dedekind rings with quotient fields $K \subseteq F$, and

$$\mathrm{Int}_D^n(A) = \{ f \in K[x_1, \ldots, x_n] \mid f(A^n) \subseteq A \}.$$

1.5. Notation and conventions. Throughout this paper, D is a domain and not a field, K the quotient field of D, and A a *torsion-free* D-algebra *finitely generated* as a D-module. Since A is faithful, there is an isomorphic copy of D embedded in A by $d \mapsto d1_A$, and we may assume $D \subseteq A$.

Now let $B = K \otimes_D A$. The natural homomorphisms $\iota_K : K \to K \otimes_D A$, $\iota_K(k) = k \otimes 1$ and $\iota_A : A \to K \otimes_D A$, $\iota_A(a) = 1 \otimes a$, are injective, since A is a torsion-free D module. We identify K and A with their isomorphic copies in B, which allows us to evaluate polynomials with coefficients in K at arguments in A, and define

$$\mathrm{Int}_D(A) = \{ f \in K[x] \mid \forall a \in A\colon\ f(a) \in A \}$$

and for $n \in \mathbb{N}_0$

$$\mathrm{Int}_D^n(A) = \{ f \in K[x_1, \dots, x_n] \mid \forall a_1, \dots, a_n \in A\colon\ f(a_1, \dots, a_n) \in A \}.$$

To exclude pathological cases we require $K \cap A = D$.

1.6. Remark. Instead of $B = K \otimes_D A$, we could look at the canonically isomorphic $A_{K \setminus \{0\}}$, the ring of fractions of A with denominators in $K \setminus \{0\}$. The natural homomorphisms $\iota_K : K \to A_{K \setminus \{0\}}$ and $\iota_A : A \to A_{K \setminus \{0\}}$ then take the form $\iota_K(\frac{c}{d}) = \frac{c1_A}{d}$ and $\iota_A(a) = \frac{a}{1}$.

1.7. Convention regarding polynomials in several variables. For non-commutative A and $n > 1$, $\mathrm{Int}_D^n(A)$ is a priori not closed under multiplication and therefore in general not a ring. With the exception of the following section on I-adic continuity, we will only consider polynomial functions in several variables if the D-algebra A is commutative.

From Section 3 onward, statements about $\mathrm{Int}_D^n(A)$ with unspecified n and A are meant as follows: if A is commutative, let $n \in \mathbb{N}_0$, if A is non-commutative, assume $n \leqslant 1$.

1.8. Remark. Note that $K \cap A = D$ implies

$$\mathrm{Int}_D(A) \subseteq \mathrm{Int}(D) = \{ f \in K[x] \mid f(D) \subseteq D \}$$

and for commutative A, also

$$\mathrm{Int}_D^n(A) \subseteq \mathrm{Int}(D^n) = \{ f \in K[x_1, \dots, x_n] \mid f(D^n) \subseteq D \},$$

and $\mathrm{Int}_D^0(A) = K \cap A = D$.

2. Continuity

I-adic continuity of the function $f : D^n \to D$ arising from a classical integer-valued polynomial $f \in \mathrm{Int}(D^n) = \{ f \in K[x_1, \dots, x_n] \mid f(D^n) \subseteq D \}$ has been shown, for an arbitrary ideal I of an arbitrary domain D, in Proposition 1.4 of [4]. (The proof there is for one variable, but clearly generalizable to several variables.)

To establish I-adic continuity of integer-valued polynomials on algebras, we will briefly look at polynomials in several non-commuting variables. If our algebra A is non-commutative, this becomes necessary, even if we are only interested in integer-valued polynomials in one variable: if we consider $f(x + y) - f(y)$ as a polynomial in two variables, and we still want substitution of elements from A for x and y to be a homomorphism, we must turn to non-commuting variables.

2.1. Definition. Let D be a domain with quotient field K. Let $K\langle x_1,\ldots,x_n\rangle$ be the free associative K-algebra generated by $x_1,\ldots,x_n$ (in other words, the semigroup-ring $K(S)$, where S is the free semigroup generated by $x_1,\ldots,x_n$). If A is a torsion-free D-algebra, we evaluate polynomials in $K\langle x_1,\ldots,x_n\rangle$ at arguments in $B=K\otimes_D A$ and thus associate a polynomial function $f: B^n\to B$ to every $f\in K\langle x_1,\ldots,x_n\rangle$. Such polynomials as map arguments in A^n to values in A we call integer-valued on A.

From the theory of PID-rings (polynomial identity rings), it is easy to garner non-trivial examples of integer-valued polynomials in several non-commuting variables. For instance, if p is prime and $n\geqslant 1$, then a polynomial in $\mathbb{Q}\langle x_1,\ldots,x_{np}\rangle$, but not in $\mathbb{Z}\langle x_1,\ldots,x_{np}\rangle$, that takes every np-tuple of $n\times n$ integer matrices to an integer matrix is

$$f(x_1,\ldots,x_{np})=\frac{1}{p}\sum_{\pi\in S_{np}} x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(np)}.$$

(This follows from [7] or [14].) In this paper, we will not consider polynomials in non-commuting variables except in the following theorem and its corollaries.

2.2. Theorem. *Let D be a domain with quotient field K and A a torsion-free D-algebra. For every $f\in K\langle x_1,\ldots,x_n\rangle$ integer-valued on A, the polynomial function $f: A^n\to A$ is uniformly I-adically continuous for every ideal I of D.*

Proof. Fix i and let d be the degree of f in x_i. We will show that for every $b\in I^dA$ and every $(a_1,\ldots,a_n)\in A^n$,

$$f(a_1,\ldots,a_i+b,\ldots,a_n)-f(a_1,\ldots,a_i,\ldots,a_n)\in IA. \qquad (*)$$

For $d=0$, or if f is the zero-polynomial, this is obvious. The polynomial f is uniquely representable as $f=f_1+f_2$, where x_i doesn't occur in f_1, x_i occurs in every monomial in the support of f_2, and f_2 has the same degree in x_i as f. Since $f_1=f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)$ and $f_2=f-f_1$, both f_1 and f_2 are integer-valued. We can show $(*)$ separately for f_1 and f_2. As $(*)$ holds for f_1 and arbitrary $b\in A$, we have reduced to the case $f=f_2$, i.e., when x_i occurs in every monomial in the support of f.

Also, it suffices to show $(*)$ for $b=t_dt_{d-1}\ldots t_1c$ with $t_k\in I$ and $c\in A$, because every element of I^dA is a finite sum of elements of this form.

By considering

$$g(x_1,\ldots,x_n,z)=f(x_1,\ldots,x_i+z,\ldots,x_n)-f(x_1,\ldots,x_i,\ldots,x_n)$$

we can reduce our task to showing for every $g(x_1,\ldots,x_n,z)\in K\langle x_1,\ldots,x_n,z\rangle$ of degree $d\geqslant 1$ in z, which maps A^{n+1} to A, satisfies $g(x_1,\ldots,x_n,0)=0$, and such that z occurs in every monomial in the support of g:

$$g(a_1,\ldots,a_n,t_d\ldots t_1c)\in IA \quad \text{whenever } t_1,\ldots,t_d\in I \text{ and } c\in A. \qquad (**)$$

We show $(**)$ by induction on d. Let $d\geqslant 1$. Consider

$$h(x_1,\ldots,x_n,y)=g(x_1,\ldots,x_n,t_dy)-t_d^dg(x_1,\ldots,x_n,y).$$

h satisfies $h(x_1,\ldots,x_n,0)=0$, is of degree at most $d-1$ in y, and y occurs in every monomial in the support of h. Now $h(a_1,\ldots,a_n,t_{d-1}\ldots t_1c)\in IA$ for all $t_1,\ldots,t_{d-1}\in I$ and $c\in A$, either by induction hypothesis or because h is the zero-polynomial. Therefore $g(a_1, \ldots, a_n, t_d t_{d-1} \ldots t_1 c) = h(a_1, \ldots, a_n, t_{d-1} \ldots t_1 c) + t_d^d g(a_1, \ldots, a_n, t_{d-1} \ldots t_1 c)$ is in IA, for all $t_1, \ldots, t_d \in I$ and $c \in A$. $\square$

Returning to commuting variables, we conclude:

2.3. Corollary. *For any ideal I of D, and every $f \in \mathrm{Int}_D^n(A)$ the function $f : A^n \to A$ is uniformly I-adically continuous.*

2.4. Corollary. *If M is a maximal ideal of D, $\hat{A}$ the M-adic completion of A, and $f \in \mathrm{Int}_D^n(A)$, then the function $f : A^n \to A$ defined by f extends uniquely to an M-adically continuous function $f : \hat{A}^n \to \hat{A}$.*

3. A few technicalities

This section contains lemmata needed for the investigation of the spectrum of $\mathrm{Int}_D(A)$ and, for commutative A, of $\mathrm{Int}_D^n(A)$. From now on, all statements about $\mathrm{Int}_D^n(A)$ are subject to the convention: if A is non-commutative, assume $n \leqslant 1$.

3.1. Lemma. *Let D be a domain and P a finitely generated prime ideal of height 1. For every non-zero $p \in P$ there exist $m \in \mathbb{N}$, and $s \in D \setminus P$ such that $sP^m \subseteq pD$.*

Proof. Let $p \in P$. In the localization D_P, P_P is the radical of (q) for every non-zero $q \in P_P$. Therefore, since P (and hence P_P) is finitely generated, there exists $m \in \mathbb{N}$ with ${P_P}^m \subseteq pD_P$ and in particular $P^m \subseteq pD_P$.

The ideal P^m is also finitely generated, by $p_1, \ldots, p_k$, say. Let $a_i \in D_P$ with $p_i = pa_i$. By considering the fractions $a_i = r_i/s_i$ (with $r_i \in D$ and $s_i \in D \setminus P$), and setting $s = s_1 \cdot \cdots \cdot s_k$, we see that $sP^m \subseteq pD$ as desired. $\square$

3.2. Definition. If $S \subseteq B^n$ and $T \subseteq B$, let

$$\mathrm{Int}_D^n(S, T) = \bigl\{ f \in K[x_1, \ldots, x_n] \bigm| f(S) \subseteq T \bigr\}.$$

3.3. Lemma. *Let D be a domain and P a finitely generated prime ideal of height 1. Then every prime ideal Q of $\mathrm{Int}_D^n(A)$ with $Q \cap D = P$ contains $\mathrm{Int}_D^n(A, PA)$.*

Proof. Let $f \in \mathrm{Int}_D^n(A, PA)$. By dint of Lemma 3.1, there are $m \in \mathbb{N}$, $p \in P$ and $s \in D \setminus P$ such that $sP^m \subseteq pD$. Then $sf^m \in \mathrm{Int}_D^n(A, pA) = p\,\mathrm{Int}_D^n(A) \subseteq Q$. As Q is prime and $s \notin Q$, we conclude that $f \in Q$. $\square$

3.4. Lemma. *Let A be a D-algebra that is finitely generated as a D-module, M a maximal ideal of finite index in D and $\hat{A}$ the M-adic completion of A. For all $a \in \hat{A}^n$, the ideal $(M\hat{A})_a = \{f \in \mathrm{Int}_D^n(A) \mid f(a) \in M\hat{A}\}$ is of finite index in $\mathrm{Int}_D^n(A)$.*

Proof. Since A is a finitely generated D-module, $\hat{A}$ is a finitely generated $\hat{D}$-module. As $\hat{M}$ is of finite index in $\hat{D}$, $\hat{A}/M\hat{A}$ is finite. Let $\mathrm{Int}_D^n(A)(a)$ denote the image of a under $\mathrm{Int}_D^n(A)$. Then $\mathrm{Int}_D^n(A)(a)/(M\hat{A} \cap \mathrm{Int}_D^n(A)(a))$, as a subring of $\hat{A}/M\hat{A}$, is finite.

Let $b_1, \ldots, b_m$ be a system of representatives of $\mathrm{Int}_D^n(A)(a)$ modulo $M\hat{A} \cap \mathrm{Int}_D^n(A)(a)$, and for $1 \leqslant i \leqslant m$ let $f_i \in \mathrm{Int}_D^n(A)$ with $f_i(a) = b_i$. Then for every $f \in \mathrm{Int}_D^n(A)$, exactly one of the differences $f - f_i$ is in $(M\hat{A})_a$, which means that $\{f_1, \ldots, f_m\}$ is a complete system of residues of $\mathrm{Int}_D^n(A)(a)$ modulo $(M\hat{A})_a$. We have shown $[\mathrm{Int}_D^n(A) : (M\hat{A})_a] = [\mathrm{Int}_D^n(A)(a) : M\hat{A} \cap \mathrm{Int}_D^n(A)(a)] \leqslant [\hat{A} : M\hat{A}]$. $\square$

4. Primes lying over a height one maximal ideal of finite index

In this section we determine the prime ideals of $\mathrm{Int}_D(A)$ (and, for commutative A, of $\mathrm{Int}_D^n(A)$) lying over a height one prime ideal of finite index of D. Prime ideals lying over a prime of infinite index in D will be characterized in the next section.

4.1. General hypotheses in this section.

(i) D is a domain and A a torsion-free D-algebra, finitely generated as a D-module;
(ii) M is a finitely generated maximal ideal of height 1 and finite index in D;
(iii) $MA_M \cap A = MA$. (Note that $MA_M \cap A = MA$ is satisfied in two important cases: if A is a free D-module, and if $D \subseteq A$ is an extension of Dedekind rings.)

We denote the M-adic completions of D and A by $\hat{D}$ and $\hat{A}$. Since M is finitely generated, M-adic and $\hat{M}$-adic topologies coincide on $\hat{D}$ and on $\hat{A}$.

4.2. Lemma. *The hypotheses of* 4.1 *imply*:

(iv) $M\hat{A} \cap A = MA$;
(v) *$\hat{D}$ and $\hat{A}$ are compact.*

Proof. (iv) Whenever M is a finitely generated maximal ideal of height 1 in a domain D and A a finitely generated D-module, the equality $M\hat{A} \cap A = MA_M \cap A$ holds, by [1, Chapter III, §2.12, Proposition 16] combined with [1, §3.5, Corollary 1 of Proposition 9].

(v) The ring $\hat{D}$ is compact because M is of finite index and finitely generated, which implies that all powers of M are of finite index. $\hat{A}$ then is compact because it is finitely generated as a $\hat{D}$-module. $\square$

4.3. Notation. Recall our convention that we only allow $n > 1$ in $\mathrm{Int}_D^n(A)$ if A is commutative; for non-commutative A, $n = 1$ is assumed.

The image of $a \in \hat{A}$ under $\mathrm{Int}_D^n(A)$ and $\mathrm{Int}_D(A)$ we denote as follows:

$$\mathrm{Int}_D^n(A)(a) = \{ f(a) \mid f \in \mathrm{Int}_D^n(A) \} \quad \text{and} \quad \mathrm{Int}_D(A)(a) = \{ f(a) \mid f \in \mathrm{Int}_D(A) \}.$$

If M is a maximal ideal of D and $a \in \hat{A}^n$ let

$$(M\hat{A})_a = \{ f \in \mathrm{Int}_D^n(A) \mid f(a) \in M\hat{A} \}.$$

If P is an ideal of a commutative ring between $\mathrm{Int}_D^n(a)$ and $\hat{A}$, let

$$P_a = \{ f \in \mathrm{Int}_D^n(A) \mid f(a) \in P \}.$$

4.4. Lemma. *Under the hypotheses of* 4.1, *let Q be a prime ideal of* $\mathrm{Int}_D^n(A)$ *lying over M. Then there exists $a \in \hat{A}$ such that*

$$(M\hat{A})_a \subseteq Q.$$

In particular, Q is of finite index.

Proof. Suppose Q does not contain any $(M\hat{A})_a$. Then for every $a \in \hat{A}^n$ there exists $f \in (M\hat{A})_a \setminus Q$. By Theorem 2.3, f is M-adically continuous, so there exists an M-adic neighborhood U of a, such that

$f(U) \subseteq M\hat{A}$. By compactness of $\hat{A}^n$, there exist finitely many a_i such that the corresponding neighborhoods U_i cover $\hat{A}^n$. Let g be the product of the polynomials $f_i \in (M\hat{A})_{a_i} \setminus Q$ with $f_i(U_i) \subseteq M\hat{A}$. For all $a \in A$, $g(a) \in M\hat{A} \cap A = MA_M \cap A = MA$ (by Lemma 4.2). Since Q is prime, $g \notin Q$, and yet $g \in \mathrm{Int}_D^n(A, MA)$, a contradiction to Lemma 3.3. We have shown that Q contains some $(M\hat{A})_a$, which is of finite index by Lemma 3.4. $\square$

In the special case $A = D$, the preceding lemma already concludes the characterization of primes of $\mathrm{Int}_D^n(D)$ lying above a maximal ideal M of finite index in D (a result of Chabert [6]), because then $(M\hat{A})_a$ is $(M\hat{D})_a = \{f \in \mathrm{Int}_D^n(D) \mid f(a) \in \hat{M}\}$, a prime ideal of finite index, and hence maximal. Chabert, however, showed the other inclusion, $Q \subseteq (M\hat{A})_a$, after first showing independently that Q must be maximal [3, Proposition V.2.2].

4.5. Lemma. *Under the hypotheses of 4.1, let Q be a prime ideal of $\mathrm{Int}_D^n(A)$ lying over M. For every $a \in \hat{A}$ such that $(M\hat{A})_a \subseteq Q$, there exists a maximal ideal P of $\mathrm{Int}_D^n(A)(a)$ such that $Q = P_a$.*

Proof. Since $\hat{A}/M\hat{A}$ is a finite ring, $\mathrm{Int}_D^n(A)(a)/(\mathrm{Int}_D^n(A)(a) \cap M\hat{A})$ is a finite commutative ring. Let $P_1, \ldots, P_k$ be the maximal ideals of $\mathrm{Int}_D^n(A)(a)$ containing $\mathrm{Int}_D^n(A)(a) \cap M\hat{A}$.

Suppose Q is not contained in any $(P_i)_a$, for $1 \leqslant i \leqslant k$. Then, by prime avoidance, $Q(a) = \{f(a) \mid f \in Q\}$ is not contained in $\bigcup_{i=1}^k P_i$. Let $f \in Q$ be such that $f(a)$ is not in any P_i. Then the residue class of $f(a)$ is a unit in $\mathrm{Int}_D^n(A)(a)/(\mathrm{Int}_D^n(A)(a) \cap M\hat{A})$.

Replacing f by a suitable power of f (using the fact that the group of units of $\mathrm{Int}_D^n(A)(a)/(\mathrm{Int}_D^n(A)(a) \cap M\hat{A})$ is finite) we see that there exists $f \in Q$ with $f(a) \equiv 1 \bmod M\hat{A}$. It follows that $1 - f \in (M\hat{A})_a \subseteq Q$ and therefore $1 \in Q$, a contradiction. $\square$

4.6. Theorem. *Let D be a domain, A a torsion-free D-algebra finitely generated as a D-module, M a finitely generated maximal ideal of D of finite index and height one, such that $MA_M \cap A = MA$.*

The prime ideals of $\mathrm{Int}_D^n(A)$ lying over M are precisely the ideals of the form

$$P_a = \bigl\{f \in \mathrm{Int}_D^n(A) \bigm| f(a) \in P\bigr\},$$

where $a \in \hat{A}$ (the M-adic completion of A), and P is a maximal ideal of $\mathrm{Int}_D^n(A)(a)$ (the image of a under $\mathrm{Int}_D^n(A)$) with $P \cap D = M$. In particular, all primes of $\mathrm{Int}_D^n(A)$ lying over M are of finite index.

Proof. There exist primes of $\mathrm{Int}_D^n(A)$ lying over M, because $(M\hat{A})_a$ with $a \in \hat{A}$ is a proper ideal of $\mathrm{Int}_D^n(A)$ containing M.

If Q is a prime ideal of $\mathrm{Int}_D^n(A)$ lying over M, then Lemma 4.4 shows that there exists an element $a \in \hat{A}$ such that $(M\hat{A})_a$ is contained in Q, and that Q is of finite index. It then follows from Lemma 4.5 that $Q = P_a$ for some maximal ideal P of $\mathrm{Int}_D^n(A)(a)$ satisfying $M \subseteq \mathrm{Int}_D^n(A)(a) \cap M\hat{A} \subseteq P$, and hence $P \cap D = M$.

Conversely, if P is a maximal ideal of $\mathrm{Int}_D^n(A)(a)$, then $\mathrm{Int}_D^n(A)/P_a$ is isomorphic to $\mathrm{Int}_D^n(A)(a)/P$. Therefore P_a is a maximal ideal of $\mathrm{Int}_D^n(A)(a)$. $\square$

It may happen that we do not know the exact image of $a \in \hat{A}$ under $\mathrm{Int}_D^n(A)$, but do know a commutative ring R_a between $\mathrm{Int}_D^n(A)(a)$ and $\hat{A}$. In this case we should remember that $R_a/(R_a \cap M\hat{A})$ is a subring of the finite ring $(\hat{A}/M\hat{A})$, and that therefore $\mathrm{Int}_D^n(A)(a)/(\mathrm{Int}_D^n(A)(a) \cap M\hat{A}) \subseteq (R_a/R_a \cap M\hat{A})$ is an extension of finite commutative rings. Since extensions of finite commutative rings satisfy "lying over", every prime ideal of $\mathrm{Int}_D^n(A)(a)$ comes from a prime ideal of R_a, and we conclude:

4.7. Corollary. *Under the hypotheses of Theorem 4.6, suppose we have, for every $a \in \hat{A}$, a commutative ring R_a with $\mathrm{Int}_D^n(A)(a) \subseteq R_a \subseteq \hat{A}$.*

Then the prime ideals of $\mathrm{Int}_D^n(A)$ *are precisely the ideals of the form*

$$P_a = \{ f \in \mathrm{Int}_D^n(A) \mid f(a) \in P \},$$

where $a \in \hat{A}$ *and* P *is a maximal ideal of* R_a *lying over* M.

If A is a commutative D-algebra, we can take $R_a = \hat{A}$ in Corollary 4.7 for all $a \in \hat{A}$, and we get the following simpler characterization of the primes of $\mathrm{Int}_D^n(A)$ lying over M:

4.8. Theorem. *Let* D *be a domain,* A *a commutative torsion-free* D*-algebra finitely generated as a* D*-module,* M *a finitely generated maximal ideal of* D *of finite index and height one, such that* $MA_M \cap A = MA$.

Then every prime ideal of $\mathrm{Int}_D^n(A)$ *lying over* M *is of the form*

$$P_a = \{ f \in \mathrm{Int}_D^n(A) \mid f(a) \in P \},$$

for some $a \in \hat{A}$ *(the* M*-adic completion of* A*) and* P *a maximal ideal of* $\hat{A}$ *lying over* M*. In particular, every prime of* $\mathrm{Int}_D^n(A)$ *lying over* M *is of finite index.*

5. Primes lying over prime ideals of infinite index

5.1. Lemma. *Let* A *be a torsion-free* D*-algebra with* $K \cap A = D$*. Let* P *be a prime ideal of infinite index in* D *and* $n \in \mathbb{N}$*. Then* $\mathrm{Int}_D^n(A) \subseteq D_P[x_1, \ldots, x_n]$.

Proof. More generally, we show that for any prime ideal P, a polynomial $f \in \mathrm{Int}_D^n(A)$ of degree less than $[D : P]$ in every individual variable is in $D_P[x_1, \ldots, x_n]$. We use induction on n. The case $n = 0$ is trivial: $\mathrm{Int}_D^0(D) = K \cap A = D \subseteq D_P$.

For $n > 0$ consider f as a polynomial in x_n with coefficients in $K[x_1, \ldots, x_{n-1}]$. Let $s \leqslant [D : P]$ such that f is of degree strictly less than s in each x_j.

Choose $d_1, \ldots, d_s \in D \subseteq A$ pairwise incongruent mod P. For every i, the value of f at d_i (substituted for x_n) is a polynomial in $\mathrm{Int}_D^{n-1}(A)$ of degree less than s in each variable. Therefore $f(x_1, \ldots, x_{n-1}, d_i) \in D_P[x_1, \ldots, x_{n-1}]$, by induction hypothesis.

Let $g \in K(x_1, \ldots, x_{n-1})[x_n]$ be the Lagrange interpolation polynomial with $g(d_i) = f(x_1, \ldots, x_{n-1}, d_i)$ $(1 \leqslant i \leqslant s)$; then $g \in D_P[x_1, \ldots, x_{n-1}][x_n]$. Since a polynomial (with coefficients in a domain) of degree less than s is determined by its values at s different arguments, we must have $f = g \in D_P[x_1, \ldots, x_{n-1}][x_n]$. $\square$

5.2. Corollary. *Let* A *be a torsion-free* D*-algebra with* $K \cap A = D$*. If all maximal ideals of* D *are of infinite index, then* $\mathrm{Int}_D^n(A) = D[x_1, \ldots, x_n]$.

Alternatively, we could have deduced the previous lemma from the corresponding fact for the ring of integer-valued polynomials over D [3, Proposition I.3.4, XI.1.10], since after all $\mathrm{Int}_D^n(A) \subseteq \mathrm{Int}(D^n) = \{f \in K[x_1, \ldots, x_n] \mid f(D^n) \subseteq D\}$.

5.3. Lemma. *Let* A *be a torsion-free* D*-algebra with* $K \cap A = D$*. Let* P *be a prime ideal of infinite index in* D *and* $n \in \mathbb{N}$*. Then the prime ideals of* $\mathrm{Int}_D^n(A)$ *lying over* P *are precisely those of the form* $Q \cap \mathrm{Int}_D^n(A)$*, where* Q *is a prime ideal of* $D_P[x_1, \ldots, x_n]$ *containing* $PD_P[x_1, \ldots, x_n]$.

Proof. As $D[x_1, \ldots, x_n] \subseteq \mathrm{Int}_D^n(A) \subseteq D_P[x_1, \ldots, x_n] = D[x_1, \ldots, x_n]_{(D \setminus P)}$, we have

$$D_P[x_1, \ldots, x_n] = \mathrm{Int}_D^n(A)_{(D \setminus P)}$$

and therefore a bijective correspondence (given by lying over) exists between prime ideals of $\mathrm{Int}_D^n(A)$ whose intersection with D is contained in P and prime ideals of $D_P[x_1,\dots,x_n]$. $\square$

5.4. Theorem. *Let D be a Noetherian one-dimensional domain with finite residue fields and A a torsion-free D algebra, finitely generated as a D-module, such that for every maximal ideal M, $MA_M \cap D = M$. If A is commutative, let $n \in \mathbb{N}$, for non-commutative A restrict to $n = 1$. Then $\mathrm{Int}_D^n(A)$ is $(n+1)$-dimensional.*

Proof. By Lemma 5.3, the prime ideals of $\mathrm{Int}_D^n(A)$ lying over (0) all come from prime ideals of $K[x_1,\dots,x_n]$. The primes of $\mathrm{Int}_D^n(A)$ lying over a maximal ideal M are all maximal and hence mutually incomparable. So $\dim(\mathrm{Int}_D^n(A)) \leqslant n+1$. If M is a maximal ideal of D and $d = (d_1,\dots,d_n) \in D^n$, let $\mathcal{Q}_k$ be the ideal of $K[x_1,\dots,x_n]$ generated by $(x_1-d_1),\dots,(x_k-d_k)$. Then a chain of primes of length $n+1$ of $\mathrm{Int}_D^n(A)$ is given by $Q_0 = (0)$, $Q_k = \mathcal{Q}_k \cap \mathrm{Int}_D^n(A)$, for $k = 1,\dots,n$ and $Q_{n+1} = P_d = \{f \in \mathrm{Int}_D^n(A) \mid f(d_1,\dots,d_n) \in P\}$, where P is a maximal ideal of the image of d under $\mathrm{Int}_D^n(A)$. $\square$

5.5. Remark. If D is a Noetherian domain of characteristic 0, finitely generated as a $\mathbb{Z}$-algebra, such as, for instance, the ring of integers $\mathcal{O}_K$ in a number field, then no maximal ideal of $\mathrm{Int}_D^n(A)$ lies over (0) of D. This holds for $A = D$ by [3, Proposition XI.3.4.], and carries over to $\mathrm{Int}_D^n(A)$, since $\mathrm{Int}_D^n(A) \subseteq \mathrm{Int}(D^n)$. Every prime ideal P of $\mathrm{Int}_D^n(A)$ coming from a prime ideal of $K[x_1,\dots,x_n]$ is contained in a maximal ideal Q of $\mathrm{Int}(D^n)$ lying over a maximal ideal M of D, and $Q \cap \mathrm{Int}_D^n(A)$ then properly contains P.

5.6. Remark. Even if there are no maximal ideals lying over (0), maximal chains of primes of $\mathrm{Int}_D^n(A)$ are not necessarily of length $n+1$. For instance, a maximal ideal of $\mathrm{Int}_D^n(A)$ may well have height 1 if it is of the form $(M\hat{A})_a$ for $a = (a_1,\dots,a_n) \in \hat{A}^n$ with $a_1,\dots,a_n$ algebraically independent over K.

6. Integer-valued polynomials on matrix algebras

Theorem 4.6 characterizes the spectrum of the ring $\mathrm{Int}_D(A)$, provided we know the images of elements of M-adic completions of A under $\mathrm{Int}_D(A)$. We will now determine these images in the case $A = M_n(D)$. Note that all the technical hypotheses in this section are certainly satisfied if $D = \mathcal{O}_K$ is the ring of integers in a number field.

6.1. Fact. (See [10, Lemma 2.2].) Let D be a domain and $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D \setminus \{0\}$. Then $f \in \mathrm{Int}_D(M_n(D))$ if and only if g is divisible modulo $dD[x]$ by all monic polynomials in $D[x]$ of degree n.

6.2. Proposition. *Let D be a domain with zero Jacobson radical and $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D \setminus \{0\}$. Then $f \in \mathrm{Int}_D(M_n(D))$ if and only if g is divisible modulo $dD[x]$ by all monic irreducible polynomials in $D[x]$ of degree n.*

Proof. In view of Fact 6.1, it suffices to show for every $d \in D \setminus \{0\}$ and $h \in D[x]$ monic of degree n, that there exists $k \in D[x]$ monic of degree n, irreducible in $D[x]$ and congruent to h mod $dD[x]$. We may choose a maximal ideal P with $d \notin P$, and use Chinese remainder theorem on the coefficients of h to find $k \in D[x]$, monic of degree n, congruent to h mod $dD[x]$ and irreducible in $(D/P)[x]$. $\square$

We are now able to identify a polynomially dense subset of $M_n(D)$ consisting of companion matrices. They are often easier to work with than general matrices, because their characteristic polynomial is also their minimal polynomial.

6.3. Theorem. *Let $\mathcal{C}_n$ be the set of companion matrices of monic polynomials of degree n in $D[x]$ and $\mathcal{I}_n \subseteq \mathcal{C}_n$ the subset of companion matrices of irreducible polynomials. If D is any domain,*

$$\mathrm{Int}_D\big(M_n(D)\big) = \mathrm{Int}_D\big(\mathcal{C}_n, M_n(D)\big).$$

If D is a domain with zero Jacobson radical, such as, for instance, a Dedekind domain with infinitely many maximal ideals, then

$$\mathrm{Int}_D\big(M_n(D)\big) = \mathrm{Int}_D\big(\mathcal{I}_n, M_n(D)\big).$$

Proof. Let $f \in \mathrm{Int}_D(\mathcal{C}_n, M_n(D))$, $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D$. Since g maps every $C \in \mathcal{C}_n$ to $M_n(dD)$, g is divisible mod $dD[x]$ by every monic polynomial in $D[x]$ of degree n. (This is so because f is still the minimal polynomial of its companion matrix when everything is viewed in D/dD.) By Fact 6.1, $f \in \mathrm{Int}_D(M_n(D))$. This shows $\mathrm{Int}_D(\mathcal{C}_n, M_n(D)) \subseteq \mathrm{Int}_D(M_n(D))$. The reverse inclusion is trivial. The argument for $\mathcal{I}_n$ is similar, using Proposition 6.2. □

6.4. Theorem. *Let D be a domain and $C \in M_n(D)$. Let*

$$\mathrm{Int}(A)(C) = \big\{f(C) \bigm| f \in \mathrm{Int}_D\big(M_n(D)\big)\big\} \quad \textit{and} \quad D[C] = \big\{f(C) \bigm| f \in D[x]\big\}.$$

Then $\mathrm{Int}(A)(C) = D[C]$.

Proof. Consider $f \in \mathrm{Int}_D(M_n(D))$; $f(x) = g(x)/d$ with $g \in D[x]$ and $d \in D \setminus \{0\}$. We know that g is divisible modulo $dD[x]$ by every monic polynomial in $D[x]$ of degree n. Dividing g by χ_C, the characteristic polynomial of C, we get

$$g(x) = q(x)\chi_C(x) + dr(x)$$

with $q, r \in D[x]$ and we see that $f(C) = r(C)$. Thus $\mathrm{Int}(A)(C) \subseteq D[C]$. The reverse inclusion is clear, since $D[x] \subseteq \mathrm{Int}_D(M_n(D))$. □

6.5. Definition. A local domain is called *analytically irreducible* if its completion is also a domain.

6.6. Lemma. *Let M be a maximal ideal of finite index in a domain D. Then the following are equivalent*

(1) $\bigcap_{n=1}^{\infty} M^n = (0)$ *and D_M is analytically irreducible*;
(2) *for every non-zero $d \in D$, cancellation of d is uniformly M-adically continuous.*

Proof. (1) ⇒ (2) We have to show: for every non-zero $d \in D$, for every $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $c \in D$: $dc \in M^k$ implies $c \in M^m$. Indirectly, suppose there exist $d \in D \setminus \{0\}$ and $m \in \mathbb{N}$, such that for every $k \in \mathbb{N}$ there is some $c_k \in D$ with $dc_k \in M^k$ and $c_k \notin M^m$. Since $\hat{D}$ is compact and satisfies first countability axiom, (c_k) has a convergent subsequence. Its limit $c \in \hat{D}$ satisfies $c \notin M^m$, and hence $c \neq 0$, and also for all k, $dc \in M^k$, which implies $dc = 0$. We have shown the existence of zero-divisors in $\hat{D}$.

(2) ⇒ (1) is easy. □

6.7. Theorem. *Let D be a domain, M a maximal ideal of finite index of D such that $\bigcap_{n=1}^{\infty} M^n = (0)$ and D_M is analytically irreducible. Let $\hat{D}$ be the M-adic completion of D, $C \in M_n(\hat{D})$, and $\mathrm{Int}_D(M_n(D))(C)$ the image of C under $\mathrm{Int}_D(M_n(D))$. Then*

$$\mathrm{Int}_D\big(M_n(D)\big)(C) \subseteq \hat{D}[C].$$

Proof. Let $f \in \mathrm{Int}_D(M_n(D))$, $f(x) = g(x)/d$ with $g \in D[x]$, $d \in D$. For every $m \in \mathbb{N}$ let $k_m \in \mathbb{N}$ such that for all $c \in D$, $cd \in M^{k_m}$ implies $c \in M^m$. Let $E_1 = (e_{ij}^{(1)})$ and $E_2 = (e_{ij}^{(2)})$ be matrices in $M_n(D)$ with characteristic polynomials χ_1 and χ_2. Then $g(x) = q_i(x)\chi_i(x) + dr_i(x)$ with $q_i, r_i \in D[x]$ for $i = 1, 2$. If $e_{ij}^{(1)} \equiv e_{ij}^{(2)} \mod M^{k_m}$, then $dr_1 \equiv dr_2 \mod M^{k_m}D[x]$, and therefore $r_1 \equiv r_2 \mod M^m D[x]$. We can therefore M-adically approximate C by matrices C_i with $f(C_i) = s_i(C_i)$ with $s_i \in D[x]$, $\deg s_i < n$, such that the s_i converge towards a polynomial $s \in \hat{D}[x]$ with $\deg s < n$ and $f(C) = s(C)$. □

6.8. Corollary. *Let D be a Dedekind domain, M a maximal ideal of finite index, $\hat{D}$ the M-adic completion of D, and $C \in M_n(\hat{D})$. Then*

$$\mathrm{Int}_D\big(M_n(D)\big)(C) \subseteq \hat{D}[C].$$

7. Integer-valued polynomials with matrix coefficients

While we have been investigating the ring $\mathrm{Int}_D(M_n(D))$ of polynomials in $K[x]$ mapping matrices in $M_n(D)$ to matrices in $M_n(D)$, Werner [13] has been studying the set, let's call it $\mathrm{Int}_D[M_n(D)]$ with square brackets, of polynomials with coefficients in the non-commutative ring $M_n(K)$ mapping matrices in $M_n(D)$ to matrices in $M_n(D)$. Without substitution homomorphism, it is not a priori clear that this set is closed under multiplication, but Werner [13] has shown that it is, and so $\mathrm{Int}_D[M_n(D)]$ is actually a ring between $\mathrm{Int}_D(\mathrm{M}_n(D))$ and $\mathrm{Int}_D(\mathrm{M}_n(K))$.

Also in [13], Werner proves that every ideal of $\mathrm{Int}_D[\mathrm{M}_n(D)]$ can be generated by elements of $K[x]$. Using the idea of his proof, one can show more: $\mathrm{Int}_D[\mathrm{M}_n(D)]$ is isomorphic to the algebra of $n \times n$ matrices over $\mathrm{Int}_D(\mathrm{M}_n(D))$. Since every prime ideal of a matrix ring is just the set of matrices with entries in a prime ideal of the ring, we get a description of the spectrum of $\mathrm{Int}_D[\mathrm{M}_n(D)]$ as a byproduct of our description of the spectrum of $\mathrm{Int}_D(\mathrm{M}_n(D))$ in the previous section. We recall the definition of prime ideal for non-commutative rings:

7.1. Definition. We call a two-sided ideal $P \neq R$ of a (not necessarily commutative) ring with identity R a prime ideal, if, for all ideals A, B of R,

$$AB \subseteq P \quad \Rightarrow \quad A \subseteq P \quad \text{or} \quad B \subseteq P,$$

or equivalently, if, for all $a, b \in R$

$$aRb \subseteq P \quad \Rightarrow \quad a \in P \quad \text{or} \quad b \in P.$$

For commutative R this is equivalent to the (in general stronger) condition: for all $a, b \in R$,

$$ab \in P \quad \Rightarrow \quad a \in P \quad \text{or} \quad b \in P.$$

7.2. Theorem. *Let D be a domain with quotient field K, and*

$$\mathrm{Int}_D\big(\mathrm{M}_n(D)\big) = \big\{f \in K[x] \bigm| \forall C \in M_n(D)\colon\ f(C) \in M_n(D)\big\},$$
$$\mathrm{Int}_D\big[\mathrm{M}_n(D)\big] = \big\{f \in \big(M_n(K)\big)[x] \bigm| \forall C \in M_n(D)\colon\ f(C) \in M_n(D)\big\}.$$

We identify $\mathrm{Int}_D[\mathrm{M}_n(D)] \subseteq (M_n(K))[x]$ *with its isomorphic image in* $M_n(K[x])$ *under*

$$\varphi : \big(M_n(K)\big)[x] \to M_n\big(K[x]\big), \quad \sum_k \big(a_{ij}^{(k)}\big)_{1 \leqslant i,j \leqslant n} x^k \mapsto \left(\sum_k a_{ij}^{(k)} x^k\right)_{1 \leqslant i,j \leqslant n}.$$

Then $\mathrm{Int}_D[\mathrm{M}_n(D)] = M_n(\mathrm{Int}_D(\mathrm{M}_n(D)))$.

Proof. Note that $K[x]$ is embedded in $M_n(K[x])$ as the subring of scalar matrices $g(x)I_n$, and in $M_n(K)[x]$ as the subring of polynomials $g(x)$ whose coefficients are scalar matrices rI_n, with $r \in K$. Clearly, $\mathrm{Int}_D[\mathrm{M}_n(D)] \cap K[x] = \mathrm{Int}_D(\mathrm{M}_n(D))$.

Let $C = (c_{ij}(x)) \in \mathrm{Int}_D[\mathrm{M}_n(D)] \subseteq M_n(K[x])$. Let e_{ij} be the matrix in $M_n(D)$ with 1 in position (i, j) and zeros elsewhere; then $e_{ij}Ce_{kl}$ has $c_{jk}(x)$ in position (i,l) and zeros elsewhere. Also, $e_{ij}Ce_{kl} \in \mathrm{Int}_D[\mathrm{M}_n(D)]$, since $\mathrm{Int}_D[\mathrm{M}_n(D)]$ is a ring containing $\mathrm{M}_n(D)$. So $\sum_{i=1}^{n} e_{ij}Ce_{ki} = c_{jk}(x)I_n \in \mathrm{Int}_D[\mathrm{M}_n(D)]$. Therefore $c_{jk}(x) \in K[x] \cap \mathrm{Int}_D[M_n(D)] = \mathrm{Int}_D(M_n(D))$ for all (j,k), and hence $\mathrm{Int}_D[\mathrm{M}_n(D)] \subseteq M_n(\mathrm{Int}_D(M_n(D)))$.

Conversely, if $f \in \mathrm{Int}_D(M_n(D))$ then $f(x)I_n \in \mathrm{Int}_D[\mathrm{M}_n(D)]$. Therefore, $e_{ik}f(x)I_ne_{kl}$, the matrix containing $f(x)$ in position (i,l) and zeros elsewhere, is in $\mathrm{Int}_D[\mathrm{M}_n(D)]$, for arbitrary (i,l). By summing matrices of this kind we see that $M_n(\mathrm{Int}_D(M_n(D))) \subseteq \mathrm{Int}_D[\mathrm{M}_n(D)]$. □

For any ring R with identity the ideals of R are in bijective correspondence with the ideals of $M_n(R)$ by $I \mapsto M_n(I)$, and restriction to prime ideals gives a bijection between the spectrum of R and the spectrum of $M_n(R)$. So we conclude:

7.3. Corollary. *Let* $\mathrm{Int}_D(M_n(D))$ *and* $\mathrm{Int}_D[M_n(D)]$ *be as in the preceding theorem. Under the identification of* $\mathrm{Int}_D[M_n(D)]$ *with its isomorphic image in* $M_n(K[x])$*,*

(1) *The two-sided ideals of* $\mathrm{Int}_D[M_n(D)]$ *are precisely the sets of the form* $M_n(I)$*, where* I *is an ideal of* $\mathrm{Int}_D(M_n(D))$.
(2) *The two-sided prime ideals of* $\mathrm{Int}_D[M_n(D)]$ *are precisely the sets of the form* $M_n(P)$*, where* P *is a prime ideal of* $\mathrm{Int}_D(M_n(D))$.

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